Bifurcation and Stability for NLS

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Lecture 1

Bifurcation: General Notions and Application to NLS

Setting of the Problem

The nonlinear Schrödinger equation

Consider the evolution equation

$$i\partial_t \psi + \Delta\psi + g(x,\psi) = 0 \qquad \text{(NLS)}$$

for $\psi = \psi(t,x) : [0,\infty) \times \mathbb{R}^N \to \mathbb{C}$, $N \geqslant 1$.

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We suppose that $g \in C(\mathbb{R}^N \times \mathbb{C}, \mathbb{R})$ satisfies

$$g(x, e^{i\theta} w) = e^{i\theta} g(x,w) \quad \text{for all } x \in \mathbb{R}^N,\ \theta \in \mathbb{R},\ w \in \mathbb{C}$$

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The coefficient $f(x, |\psi|^2)$ enables one to model the response of a nonlinear inhomogeneous medium to some field ψ.

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Some famous names associated with NLS: Ginibre, Velo, Cazenave, Lions, Merle, Raphaël, Gérard, Weinstein, Bourgain, Strauss, Tao, Vega, Ambrosetti, Malchiodi, etc.

Standing waves of NLS

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The equivariance of g with respect to the group $\{e^{i\theta} : \theta \in \mathbb{R}\}$ allows one to look for standing wave solutions of the form :

$$\psi(t,x) = e^{i\lambda t} u(x) \quad \text{with} \quad \lambda \in \mathbb{R} \quad \text{and} \quad u \in H^1(\mathbb{R}^N, \mathbb{R})$$

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Notice that we have a line of trivial solutions $\{(\lambda, 0) : \lambda \in \mathbb{R}\}$.

Bifurcation Theory

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Definition 1

$\lambda \in \mathbb{R}$ is a bifurcation point for problem (P) iff there is a sequence $\{(\lambda_n, u_n)\} \subset \mathbb{R} \times X$ such that $\lambda_n \to \lambda$ and $u_n \to 0$ as $n \to \infty$, but $u_n \neq 0$ for all n.

Linearization, spectral analysis

$$F(u) = \lambda u, \quad \lambda \in \mathbb{R} \tag{P}$$

If in addition we suppose that $F \in C^1(X, Y)$, it is natural to approximate (P) in a neighbourhood of $u = 0$ by the linearized problem

$$DF(0)u = \lambda u \tag{LP}$$

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By the implicit function theorem, all bifurcation points belong to $\sigma(DF(0))$, the spectrum of $DF(0)$.

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$$\sigma(DF(0)) = \sigma_{\text{disc}}(DF(0)) \cup \sigma_{\text{ess}}(DF(0))$$

where $\sigma_{\text{disc}}(DF(0))$ is the discrete spectrum, i.e. the set of finite multiplicity eigenvalues that are isolated in $\sigma(DF(0))$, and

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Bifurcation can thus occur from an eigenvalue of $DF(0)$, or from the essential spectrum of $DF(0)$.

Schrödinger operators

Let $Y = L^2(\mathbb{R}^N)$, $X = H^2(\mathbb{R}^N)$ and $F : X \to Y$ defined by

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Then $F \in C^1(X, Y)$ and the linearization of (SNLS) is given by

$$\Delta u + q(x)u = \lambda u$$

The operator $S : D(S) = H^2(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ defined by

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Furthermore, Λ is given by the explicit formula

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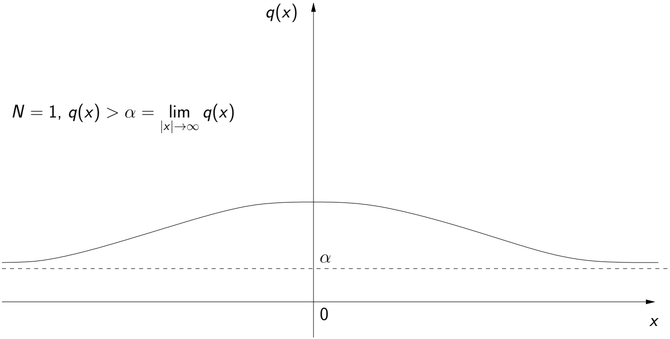
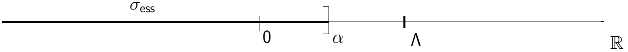
If $\Lambda > \alpha$ then $\Lambda \in \sigma_{\mathrm{disc}}(S)$. In this case, Λ is the principal eigenvalue of S, and S has a corresponding eigenfunction $\varphi_\Lambda > 0$.



$\sigma\left(\frac{d^2}{dx^2} + q(x)\right)$



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(C) Moreover in 1971, Crandall and Rabinowitz showed that, locally, the branch bifurcating from a simple eigenvalue is a continuous curve.

(A) and (B) used the Leray-Schauder degree, which requires some form of compactness. These results have later been extended to non-compact cases. Result (C) is purely analytical, and more flexible in that respect.

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As shall be seen later on, smooth curves of positive solutions bifurcating from the essential spectrum can be obtained by analytical methods (cf. Stuart 1985, Stuart-G. 2008, G. 2009).

Examples

Bifurcation from the principal eigenvalue

Consider the one-dimensional problem

$$\begin{cases} u''(x) + g(x, u(x)) = \lambda u(x), & x \in \mathbb{R} \\ \lim_{|x| \to \infty} u(x) = 0 \end{cases} \qquad \text{(E1)}$$

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As earlier, we consider the problem in $X = H^2(\mathbb{R})$.

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As earlier, we consider the problem in $X = H^2(\mathbb{R})$.

$\Lambda = \sup \sigma(S) > 0 = \sup \sigma_{\text{ess}}(S)$ is the principal eigenvalue of S.

Example

A typical example of a function g satisfying the above hypotheses is given by

$$g(x, s) = q(x)s + V(x)|s|^{p-1}s$$

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Theorem 1 (Jeanjean-Stuart, Adv. Diff. Equ. 1999)

There exist $\bar{\lambda} \in (\Lambda, \infty]$ *and a curve* $u \in C^1\big((\Lambda, \bar{\lambda}), H^2(\mathbb{R})\big)$ *such that* $(\lambda, u(\lambda))$ *is a solution of* (E1) *for all* $\lambda \in (\Lambda, \bar{\lambda})$*, with* $u(\lambda)(x)$ *positive, even, and strictly decreasing in* $x > 0$*.*

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Furthermore,

$$\lim_{\lambda \searrow \Lambda} \|u(\lambda)\|_{H^2} = 0 \quad \textit{and} \quad \lim_{\lambda \nearrow \bar{\lambda}} \|u(\lambda)\|_{H^2} = \infty$$

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By topological arguments, (global) connected sets of solutions can be obtained under weaker assumptions, see for instance Jeanjean-Lucia-Stuart 1999.

The hypothesis $q(0) > \lim_{|x|\to\infty} q(x)$ ensures that Λ is a (simple) eigenvalue. Without this assumption, we can consider bifurcation from the essential spectrum of S.

Lecture 2

Bifurcation from the Essential Spectrum

The Power Nonlinearity

Consider the problem

$$\begin{cases} \Delta u(x) + V(x)|u(x)|^{p-1}u(x) = \lambda u(x), \quad x \in \mathbb{R}^N \\ \lim_{|x|\to\infty} u(x) = 0 \end{cases} \tag{E2}$$

for $N \geqslant 2$.

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(V1) $V \in C^1(\mathbb{R}^N)$

(V2) there exists $b \in (0,2)$ such that

$$1 < p < \frac{4-2b}{N-2} \ (:= \infty \text{ if } N = 2)$$

$$\lim_{|x|\to\infty} |x|^b V(x) = 1 \quad \text{and} \quad \lim_{|x|\to\infty} |x|^b [x \cdot \nabla V(x) + bV(x)] = 0$$

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(V4) $r\dfrac{V'(r)}{V(r)}$ is decreasing in $r > 0$ (and so $\searrow -b$ by (V2))

Example

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Thus

$$S = \Delta \quad \text{and} \quad \sigma(S) = \sigma_{\text{ess}}(S) = (-\infty, 0]$$

and so S has no eigenvalues.

Nevertheless, the following result holds in $X = H^1(\mathbb{R}^N)$.

Theorem 2 (Stuart-G., DCDS 2008)

We suppose (V1) and (V2).

There exist $\lambda_0 > 0$ *and a local curve* $u \in C^1\big((0,\lambda_0), H^1(\mathbb{R}^N)\big)$ *such that* $(\lambda, u(\lambda))$ *is a solution of* (E2) *for all* $\lambda \in (0,\lambda_0)$*, with* $u(\lambda) \in C^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *and* $u(\lambda) > 0$ *on* $\mathbb{R}^N$*.*

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Furthermore,

$$\lim_{\lambda \searrow 0} \|u(\lambda)\|_{H^1} = \begin{cases} 0 & \text{if} \quad 1 < p < 1 + \frac{4-2b}{N} \\ \infty & \text{if} \quad 1 + \frac{4-2b}{N} < p < 1 + \frac{4-2b}{N-2} \end{cases}$$

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Terminology

We say that there is bifurcation from the line of trivial solutions when $\|u(\lambda)\|_{H^1} \to 0$ and asymptotic bifurcation, or bifurcation from infinity, when $\|u(\lambda)\|_{H^1} \to \infty$.

Theorem 3 (G., Calc. Var. 2010)

Suppose (V1) to (V4).

Then the curve of Theorem 2 can be extended to a global branch $u \in C^1\big((0,\infty), H^1(\mathbb{R}^N)\big)$ *such that, for all* $\lambda \in (0,\infty)$:

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Application

The case $N = 1$, $p = 3$ yields existence of travelling waves in self-focusing planar waveguides with ‘Kerr materials’, for arbitrary high/low power beams, see G., Adv. Nonlin. Stud. 2010.

Proofs

We start by the scaling

$$\lambda = k^2, \quad u(x) = k^\theta v(y), \quad y = kx, \quad \text{for } k > 0, \quad \theta = \tfrac{2-b}{p-1}$$

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suggesting the limit problem

$$\Delta v - v + |y|^{-b} |v|^{p-1} v = 0 \tag{2}$$

which has a unique positive radial solution $v_0 \in H^1(\mathbb{R}^N)$.

We then apply the implicit function theorem to the function $F : \mathbb{R} \times H^1(\mathbb{R}^N) \to H^{-1}(\mathbb{R}^N)$ defined by

$$F(k,v) := \begin{cases} \Delta v - v + |k|^{-b} V(y/|k|)|v|^{p-1}v, & k \neq 0 \\ \Delta v - v + |y|^{-b}|v|^{p-1}v, & k = 0 \end{cases}$$

at the point $(k,v) = (0, v_0) \in \mathbb{R} \times H^1(\mathbb{R}^N)$, where $D_2F(0,v_0) : H^1 \to H^{-1}$ is an isomorphism (non-degeneracy), which yields a branch of solutions $(k, v(k))$ to $F(k,v) = 0$ (with $|k| < k_0$ small).

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Going back to the original variables (λ, u), we then get a local branch of solutions $(\lambda, u(\lambda))$ of (E2), for all $0 < \lambda < \lambda_0 = k_0^2$. The asymptotic behaviour as $\lambda \to 0$ follows from the change of variables, using $v(k) \to v_0$ in H^1 as $k \to 0$.

To prove Theorem 3, one shows that the local branch given by Theorem 2 can be extended indefinitely to a C^1 curve parametrized by $\lambda > 0$.

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It can then be shown that u_λ is a non-degenerate solution, for all $\lambda > 0$. The IFT can thus be applied to each point (λ, u_λ), and one finally concludes that the branch started in Theorem 2 extends indefinitely.

Remarks

This method is purely analytical and yields strong conclusions : smooth curves, precise asymptotic behaviour and monotonicity of the branch, etc. These properties will be useful to study the stability of standing waves of (NLS).

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Purely variational arguments (see e.g. Stuart '82 '88) yield sequences of solutions converging to the line of trivial solutions (this is the weakest possible notion of bifurcation, corresponding exactly to Definition 1).

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Purely variational arguments (see e.g. Stuart '82 '88) yield sequences of solutions converging to the line of trivial solutions (this is the weakest possible notion of bifurcation, corresponding exactly to Definition 1).

On the other hand, topological methods yield connected sets of solutions (which are not necessarily arcwise connected), see for instance Toland '82, Giacomoni '98.

Bifurcation from the Essential Spectrum for More General Nonlinearities

A perturbative approach

Theorem 2 can be extended to more general nonlinearities than $V(x)|u|^{p-1}u$ by using a perturbative method.

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Consider the problem

$$\begin{cases} \Delta u + g(x,u) = \lambda u, & x \in \mathbb{R}^N \\ \lim\limits_{|x|\to\infty} u(x) = 0 \end{cases} \tag{E3}$$

for $N \geqslant 2$, with

$$g(x,s) = V(x)|s|^{p-1}s + r(x,s)$$

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Then Theorem 2 still holds for (E3), cf. G., JDE 2009.

Example

This approach allows one to handle a sum of powers

$$g(x,s) = V(x)|s|^{p-1}s + \sum_{i=1}^{m} Z_i(x)|s|^{q_i-1}s$$

if, for $i = 1, \ldots, m$, there holds

$$Z_i \in C^1(\mathbb{R}^N), \quad |x|^b Z_i(x) \text{ is bounded}, \quad \text{and } q_i > p$$

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It also covers the asymptotically linear case

$$g(x,s) = V(x)\frac{|s|^{p-1}}{1+|s|^{p-1}}s \qquad (\star)$$

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that we will now study thoroughly. Hence, we already know that there is (local) bifurcation from the line of trivial solutions for NLS with the nonlinearity $(\star)$, provided $1 < p < 1 + \frac{4-2b}{N}$.

Asymptotic Bifurcation for the Asymptotically Linear NLS

The asymptotically linear NLS

$$i\partial_t \psi + \Delta\psi + f(x, |\psi|^2)\psi = 0 \qquad \text{(NLS)}$$

$$\Delta u + f(x, u^2)u = \lambda u, \quad u \in H^1(\mathbb{R}^N) \qquad \text{(SNLS)}$$

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For simplicity, we suppose that f has the form

$$f(x, s^2) = V(x)\frac{s^{p-1}}{1 + s^{p-1}}, \quad x \in \mathbb{R}^N,\ s \geqslant 0$$

for some $V \in C(\mathbb{R}^N, \mathbb{R}_+) \cap L^\infty(\mathbb{R}^N, \mathbb{R}_+)$ and $p > 1$.

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Then the equation (NLS) is termed asymptotically linear :

$$f(x, s^2) \to V(x) \quad \text{as} \quad s \to \infty$$

Applications : nonlinear waveguides with saturable refractive index.

Asymptotic linearization

We call asymptotic linearization of (SNLS) the equation

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and suppose that

$$\lambda_\infty := -\inf_{\substack{u\in H^1(\mathbb{R}^N)\\ u\neq 0}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 - V(x)u^2\,\mathrm{d}x}{\int_{\mathbb{R}^N} u^2\,\mathrm{d}x} > \lambda_*$$

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Then $\qquad \sigma_{\mathrm{ess}}(\mathrm{AL}) \subset (-\infty, \lambda_*]$

Asymptotic linearization

We call asymptotic linearization of (SNLS) the equation

$$\Delta u + V(x)u = \lambda u, \quad u \in H^1(\mathbb{R}^N) \qquad \text{(AL)}$$

Let

$$\lambda_* := \limsup_{|x|\to\infty} V(x) \in [0, \infty)$$

and suppose that

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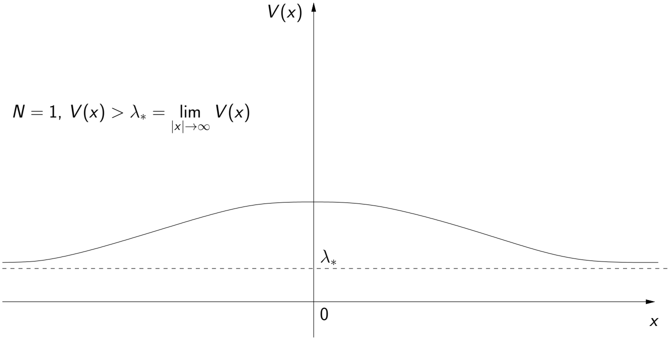
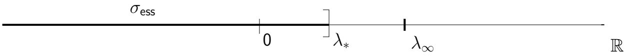
Then $\sigma_{\text{ess}}(\text{AL}) \subset (-\infty, \lambda_*]$

and λ_∞ is the principal eigenvalue of (AL).



$\sigma\left(\frac{d^2}{dx^2} + V(x)\right)$



Inversion

Let $X := W^{2,q}(\mathbb{R}^N)$, $Y := L^q(\mathbb{R}^N)$, $q \in [2, \infty) \cap (\frac{N}{2}, \infty)$, and $\| \cdot \|$ the usual norm of X.

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Now bifurcation for $(\star)$ from $v = 0$ at $\lambda = \lambda_\infty$ (the principal eigenvalue of (AL)) is in principle equivalent to asymptotic bifurcation for (SNLS) at $\lambda = \lambda_\infty$.

Truncation

The mapping

$$v \mapsto \begin{cases} h(\cdot, v/\|v\|^2)v, & v \neq 0 \\ 0, & v = 0 \end{cases}$$

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Nevertheless, for all $n \in \mathbb{N}$, the truncated problem

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Furthermore, $v \mapsto \chi_{\{|\cdot| \leqslant n\}} h(\cdot, v/\|v\|^2)v$ is compact.

Global bifurcation

On the other hand,

$$(\lambda_*, \infty) \ni \lambda \mapsto L(\lambda) := \Delta + V - \lambda$$

defines a C^1 family of Fredholm operators of index 0, such that

$$L'(\lambda_\infty) \ker L(\lambda_\infty) \oplus \operatorname{rge} L(\lambda_\infty) = Y \quad \text{and} \quad \dim \ker L(\lambda_\infty) = 1$$

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A global bifurcation theorem of Stuart & Zhou (2006) can then be applied, yielding a connected set $\mathcal{C}_n \subset \mathbb{R} \times X$ of positive solutions of $(\star_n)$, bifurcating from the point $(\lambda_\infty, 0)$, for all $n \in \mathbb{N}$.

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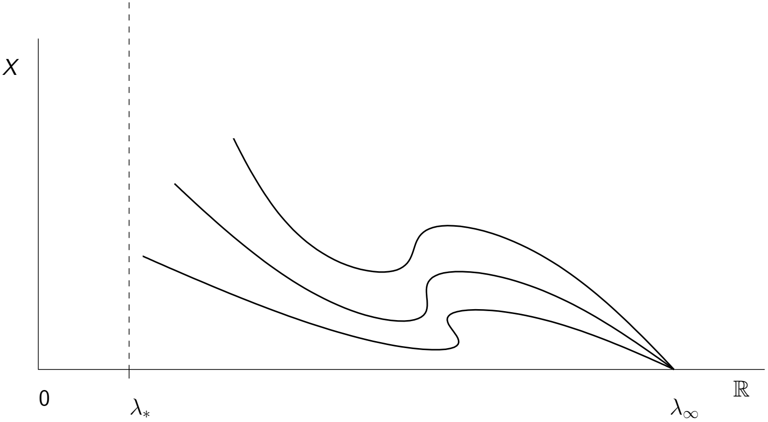
This global result is based on a topological degree of Rabier & Salter (2005), dealing with compact perturbations of Fredholm operators of index 0.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There exists a continuum $\mathcal{S} \subset \mathbb{R} \times X$ of positive solutions of (SNLS) *with the following properties*:

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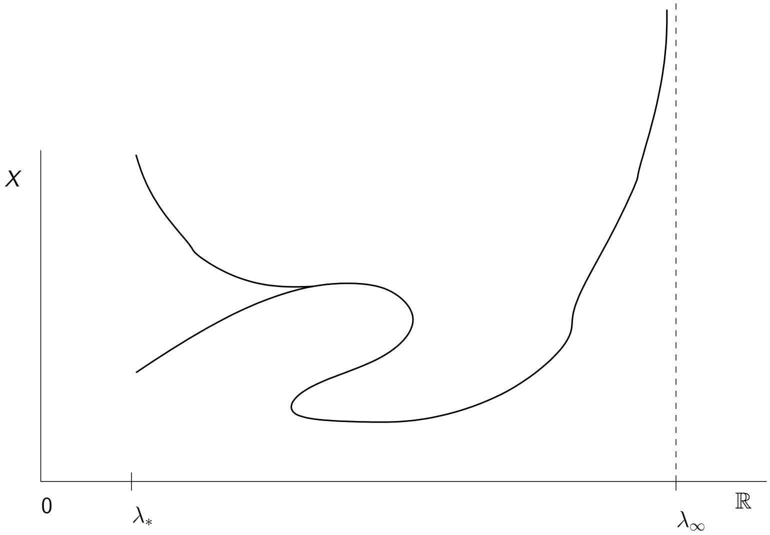
(iii) *For any* $\{(\lambda_n, u_n)\} \subset \mathcal{S}$ *such that* $\lambda_n \to \lambda$ *as* $n \to \infty$, *there holds*

$$\lim_{n\to\infty} \|u_n\|_{L^q(\mathbb{R}^N)} = \lim_{n\to\infty} \|u_n\|_{L^\infty(\mathbb{R}^N)} = \infty \iff \lambda = \lambda_\infty$$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Under stronger assumptions, in dimension $N = 1$, we will now show that global bifurcation actually occurs along a continuous (even smooth) curve of solutions.

The 1-dimensional Case

A continuous curve

$$u'' + f(x,u^2)u = \lambda u, \quad u \in H^1(\mathbb{R}) \qquad \text{(SNLS)}$$

$$f(x,s^2) = V(x)\frac{s^{p-1}}{1+s^{p-1}}$$

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If we suppose V even and decreasing on $[0,\infty)$, then the positive solution $(\lambda, u_\lambda) \in \mathcal{S}$ is unique, for all $\lambda \in (\lambda_*, \lambda_\infty)$, and satifies

$$u_\lambda \text{ even and } u'_\lambda < 0 \text{ on } (0,\infty).$$

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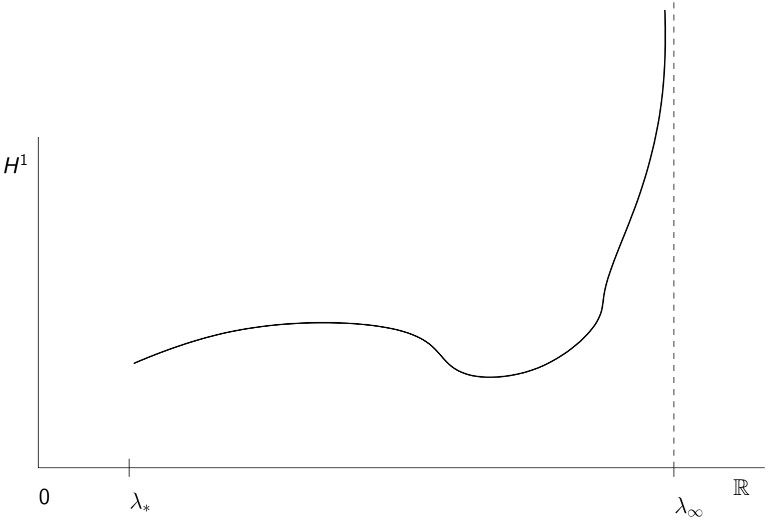
A compactness argument then shows that

$$\mathcal{S} = \{(\lambda, u_\lambda) : \lambda \in (\lambda_*, \lambda_\infty)\}$$

is a continuous curve.


H^1
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Continuation down to $u = 0$

Let us further suppose that

$$V(x) \sim |x|^{-b} \quad \text{as} \quad |x| \to \infty \quad \text{for some } b \in (0,1)$$

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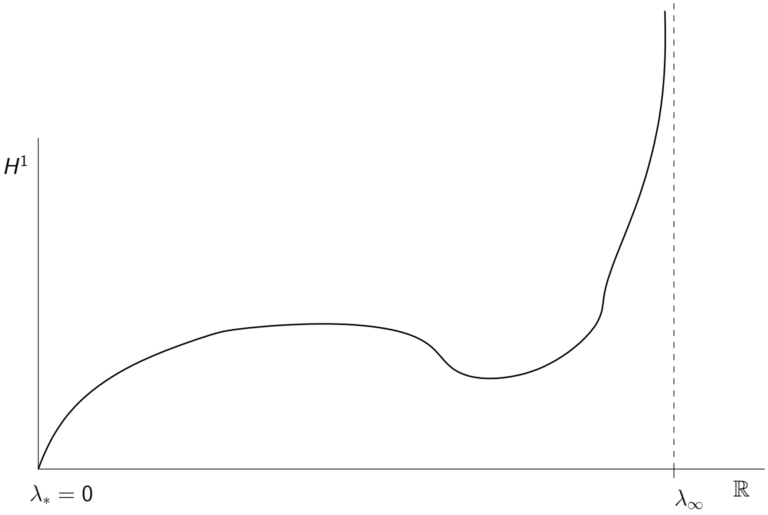
We thus have bifurcation from a point of the essentiel spectrum of the linearization

$$u'' = \lambda u$$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Non-degeneracy

If we assume, in addition, that V is C^1 and $V' < 0$ on $(0, \infty)$, then the linearized operator $T_\lambda : H^1(\mathbb{R}) \to H^{-1}(\mathbb{R})$,

$$T_\lambda v := v'' + [f(x, u_\lambda^2) + 2\partial_2 f(x, u_\lambda^2) u_\lambda^2] v - \lambda v$$

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Hence, applying the implicit function theorem to each point (λ, u_λ), we see that, in fact,

$$(0, \lambda_\infty) \ni \lambda \mapsto u_\lambda \in H^1(\mathbb{R}) \text{ is a } C^1 \text{ map}$$

Theorem 5 (G., EECT 2013)

Suppose that $V \in C^1(\mathbb{R})$ *is even,* $V' < 0$ *on* $(0, \infty)$*, and that*

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In fact, $u_\lambda \in C^2(\mathbb{R}) \cap H^2(\mathbb{R})$ *with* $u_\lambda' < 0$ *on* $(0,\infty)$, *and* $u_\lambda, u_\lambda' \to 0$ *exponentially as* $|x| \to \infty$.

Lecture 3

Stability of Standing Waves

The 1-dimensional evolution equation

Let us now go back to the time-dependent equation

$$i\partial_t \psi + \partial^2_{xx}\psi + f(x, |\psi|^2)\psi = 0 \qquad \text{(NLS)}$$

for

(PP) $f(x, s^2) = V(x)s^{p-1}$ or (AL) $f(x, s^2) = V(x)\dfrac{s^{p-1}}{1 + s^{p-1}}$

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We also suppose that $1 < p < 5 - 2b$ ('subcritical' nonlinearity).

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Under these hypotheses, we found solutions

$$\psi_\lambda(t,x) := e^{i\lambda t} u_\lambda(x), \quad 0 < \lambda < \lambda_\infty \begin{cases} = \infty \text{ (PP)} \\ < \infty \text{ (AL)} \end{cases}$$

where $u_\lambda \in H^2(\mathbb{R}) \cap C^2(\mathbb{R})$ satisfies the stationary problem

$$\begin{cases} u''(x) + f(x, u(x)^2)u(x) = \lambda u(x), \quad x \in \mathbb{R} \\ \lim\limits_{|x|\to\infty} u(x) = 0 \end{cases} \tag{SNLS}$$

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$u(x) \equiv u(-x) > 0$ and $\lambda \mapsto u_\lambda$ is $C^1\big((0,\lambda_\infty), H^1(\mathbb{R})\big)$

with $\|u_\lambda\|_{H^1} \to 0$ as $\lambda \to 0$.

Orbital stability

Since (NLS) is invariant under the action of the group $\{e^{i\theta} : \theta \in \mathbb{R}\}$, one cannot expect the periodic solutions $\psi_\lambda(t,x) = e^{i\lambda t} u_\lambda(x)$ to be stable in the usual sense.

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Indeed, suppose $\lambda_n \to \lambda \in (0, \lambda_0)$ and consider

$$\psi_\lambda(t,x) = e^{i\lambda t} u_\lambda(x) \quad \text{and} \quad \varphi_n(t,x) = e^{i\lambda_n t} u_{\lambda_n}(x)$$

Then $\forall\, \delta > 0\ \exists\, N_\delta \in \mathbb{N}$ s.t.

$$n \geqslant N_\delta \implies \|\varphi_n(0,\cdot) - \psi_\lambda(0,\cdot)\|_{H^1} = \|u_{\lambda_n} - u_\lambda\|_{H^1} \leqslant \delta$$

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However,

$$\|\varphi_n(t,\cdot) - \psi_\lambda(t,\cdot)\|_{H^1} \geqslant \Big| |e^{i\lambda t} - e^{i\lambda_n t}| \|u_\lambda\|_{H^1} - \|u_{\lambda_n} - u_\lambda\|_{H^1} \Big|$$

$$\implies \sup_{t \geqslant 0} \|\varphi_n(t) - \psi_\lambda(t)\|_{H^1} \geqslant 2\|u_\lambda\|_{H^1} - \delta \quad \text{for } n \geqslant N_\delta$$

The notion of stability which is naturally suited to periodic solutions is the following.

Definition 2

A standing wave $\psi(t,x) = \mathrm{e}^{i\lambda t} u(x)$ is called orbitally stable iff $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ s.t. for any solution φ of (NLS) there holds

$$\|\varphi(0,\cdot) - u\|_{H^1} \leqslant \delta \implies \inf_{\theta \in \mathbb{R}} \|\varphi(t,\cdot) - \mathrm{e}^{i\theta} u\|_{H^1} \leqslant \varepsilon \quad \forall\, t \geqslant 0$$

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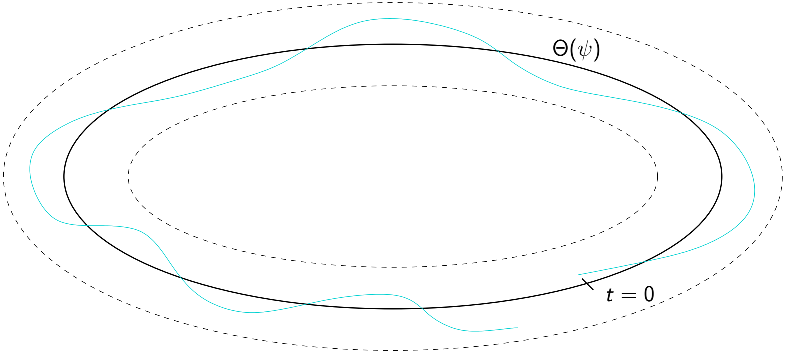
Intuitively: $\varphi(0,\cdot)$ close to $u \implies$

$\varphi(t,\cdot)$ close to the orbit $\Theta(\psi) := \{\mathrm{e}^{i\theta} u : \theta \in \mathbb{R}\}\ \forall\, t \geqslant 0.$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For any given $\lambda_0 \in (0, \lambda_\infty)$, the stability/instability of the standing wave $\psi_{\lambda_0} = e^{i\lambda_0 t} u_{\lambda_0}$ can be discussed by the following conditions.

(1) Interpreting (NLS) as a Hamiltonian system, spectral conditions ensure that the Hessian of the system at ψ_{λ_0} has only one, potentially, unstable direction.

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(1) Interpreting (NLS) as a Hamiltonian system, spectral conditions ensure that the Hessian of the system at ψ_{λ_0} has only one, potentially, unstable direction.

(2) Then the slope condition asserts that :

$$\psi_{\lambda_0} \text{ is stable/unstable if}$$

$$\lambda \mapsto \|u_\lambda\|_{L^2} \text{ is strictly increasing/decreasing at } \lambda = \lambda_0$$

In fact, if it is increasing then the Hessian has no real eigenvalues, the system is linearly stable and a Lyapunov function can be constructed to show (nonlinear) orbital stability. If it is decreasing then the Hessian has a positive eigenvalue and the system is (linearly) unstable.

The spectral conditions

For $\lambda \in (0, \lambda_\infty)$, consider the linear operators $L_\lambda^+, L_\lambda^- : H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined by

$$L_\lambda^+ v = -v'' + \lambda v - [f(x, u_\lambda^2) + 2\partial_2 f(x, u_\lambda^2) u_\lambda^2] v$$
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The spectral conditions required by the stability analysis are :

(S1) $\inf \sigma_{\text{ess}}(L_\lambda^+) > 0, \quad M(L_\lambda^+) = 1, \quad \ker L_\lambda^+ = \{0\}$

(S2) $\inf \sigma_{\text{ess}}(L_\lambda^-) > 0, \quad 0 = \inf \sigma(L_\lambda^-), \quad \ker L_\lambda^- = \text{vect}\{u_\lambda\}$

where $M(L_\lambda^+)$ is the Morse index of L_λ^+, i.e. the dimension of the larger subspace where L_λ^+ is negative definite.

$$L_\lambda^+ v = -v'' + \lambda v - [f(x, u_\lambda^2) + 2\partial_2 f(x, u_\lambda^2) u_\lambda^2] v$$
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Verification

First, all the eigenvalues are simple.

$$L_\lambda^+ v = -v'' + \lambda v - [f(x, u_\lambda^2) + 2\partial_2 f(x, u_\lambda^2) u_\lambda^2] v$$
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It remains to show that L_λ^+ has exactly one negative eigenvalue.

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The slope condition

We will check that the function $\lambda \mapsto \|u_\lambda\|_{L^2}$ is strictly increasing on $(0, \lambda_\infty)$.

Since $\|u_\lambda\|_{L^2} \to 0$ as $\lambda \to 0$ thanks to Theorems 2/5, this is true in a neighbourhood of some $\lambda > 0$. Hence we need only verify that

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First notice that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \int_{\mathbb{R}} u_\lambda^2 \,\mathrm{d}x = 2 \int_{\mathbb{R}} u_\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} u_\lambda \,\mathrm{d}x = 4 \int_0^\infty u_\lambda \xi_\lambda$$

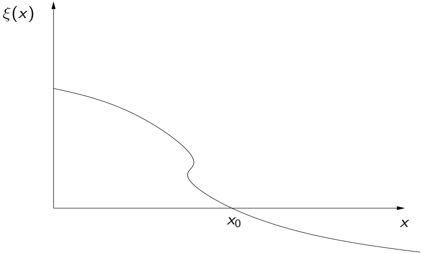
where $\xi_\lambda := \dfrac{\mathrm{d}}{\mathrm{d}\lambda} u_\lambda$ satisfies

$$\xi_\lambda'' + [f(x, u_\lambda^2) + 2\partial_2 f(x, u_\lambda^2) u_\lambda^2]\, \xi_\lambda = \lambda \xi_\lambda + u_\lambda$$

It can be shown that

$$\int_0^\infty [2f(x,u^2) + x\partial_1 f(x,u^2) - \partial_2 f(x,u^2)u^2]u\xi \,\mathrm{d}x = 2\lambda \int_0^\infty u\xi \,\mathrm{d}x \qquad (\star)$$

and that ξ has this profil :



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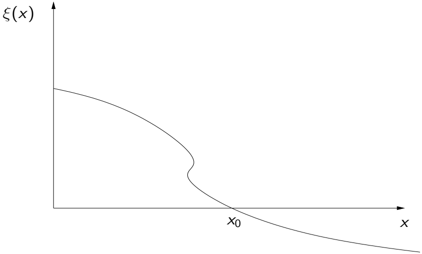

Supposing by contradiction that $\int_0^\infty u\,\xi\,\mathrm{d}x = 0$, we can write $(\star)$ as

$$\int_0^\infty \Big[\frac{2f(x,u^2) + x\partial_1 f(x,u^2)}{\partial_2 f(x,u^2)u^2} - 1\Big]\partial_2 f(x,u^2)u^3\xi \,\mathrm{d}x = 0$$

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Defining $\zeta(x) := \frac{2f(x,u^2)+x\partial_1 f(x,u^2)}{\partial_2 f(x,u^2)u^2} - 1$, this becomes

$$\int_0^\infty \zeta(x)\partial_2 f(x,u^2)u^3\xi \, \mathrm{d}x = 0$$

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Now using the unique zero x_0 of ξ, we can rewrite this identity as

$$\int_0^\infty [\zeta(x)-\zeta(x_0)]\partial_2 f(x,u^2)u^3\xi \,\mathrm{d}x + \zeta(x_0)\int_0^\infty \partial_2 f(x,u^2)u^3\xi \,\mathrm{d}x = 0$$

$$\int_0^\infty [\zeta(x) - \zeta(x_0)]\partial_2 f(x, u^2)u^3\xi \, \mathrm{d}x + \zeta(x_0) \int_0^\infty \partial_2 f(x, u^2)u^3\xi \, \mathrm{d}x = 0$$

Moreover, the Lagrange identity for u and ξ yields

$$\int_0^\infty u^2 \, \mathrm{d}x = 2 \int_0^\infty \partial_2 f(x, u^2)u^3\xi \, \mathrm{d}x$$

and so

$$\int_0^\infty \partial_2 f(x, u^2)u^3[\zeta(x) - \zeta(x_0)]\xi \, \mathrm{d}x + \frac{\zeta(x_0)}{2} \int_0^\infty u^2 \, \mathrm{d}x = 0 \quad (\star\star)$$

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Now,

$$\partial_2 f(x,u^2)u^3 = \begin{cases} \frac{p-1}{2}V(x)u^p & \text{in the (PP) case} \\ \frac{p-1}{2}V(x)\frac{u^p}{(1+u^{p-1})^2} & \text{in the (AL) case} \end{cases}$$

hence $\partial_2 f(x,u^2)u^3 > 0$ on $(0,\infty)$ in any case.

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hence $\partial_2 f(x,u^2)u^3 > 0$ on $(0,\infty)$ in any case.

On the other hand,

$$\zeta(x) = \begin{cases} \frac{2}{p-1}[x\frac{V'(x)}{V(x)} + \frac{5-p}{2}] & \text{(PP)} \\ \frac{2}{p-1}[x\frac{V'(x)}{V(x)} + \frac{5-p}{2}] + \frac{2}{p-1}[x\frac{V'(x)}{V(x)} + 2]u^{p-1} & \text{(AL)} \end{cases}$$

and we will see that $\zeta > 0$ and $\searrow$ in any case, contradicting $(\star\star)$.

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Indeed, using the hypotheses

$$x \mapsto x\frac{V'(x)}{V(x)} \searrow, \quad x\frac{V'(x)}{V(x)} \geqslant -b \quad \text{and} \quad p < 5 - 2b$$

$$\text{we have} \quad x\frac{V'(x)}{V(x)} + \frac{5-p}{2} > 0 \ \text{ and } \ \searrow$$

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$$\text{we have} \quad x\frac{V'(x)}{V(x)} + \frac{5-p}{2} > 0 \ \text{ and } \ \searrow$$

Furthermore,

$$u > 0 \ \text{ and } \ \searrow \implies \underbrace{\left[x\frac{V'(x)}{V(x)} + 2\right]}_{\geqslant -b+2>0} u^{p-1} > 0 \ \text{ and } \ \searrow$$

so that $\zeta > 0$ and $\searrow$ in any case, as expected.

Theorem 6 (G., ANS 2009/EECT 2013)

Suppose that $V \in C^1(\mathbb{R})$ *is even,* $V' < 0$ *on* $(0, \infty)$*, and that*

$$V(x) \sim |x|^{-b} \quad \textit{as} \quad |x| \to \infty \quad \textit{for some } b \in (0,1)$$

with $1 < p < 5 - 2b$*. Suppose in addition that*

$$x \mapsto x\frac{V'(x)}{V(x)} \textit{ is decreasing on } (0,\infty) \textit{ with } x\frac{V'(x)}{V(x)} \searrow -b$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\int_{\mathbb{R}} u_\lambda^2 \,\mathrm{d}x > 0 \quad \forall\, \lambda \in (0, \lambda_\infty)$$

In particular, the standing wave $\psi_\lambda(t,x) = \mathrm{e}^{i\lambda t}u_\lambda(x)$ *is an orbitally stable solution of* (NLS) *for all* $\lambda \in (0, \lambda_\infty)$.