Optical solitons in nonlinear waveguides

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Part I: Nonlinear waves in planar waveguides

The inhomogeneous nonlinear Schrödinger equation

I will present a derivation of the nonlinear Schrödinger equation

$$i\partial_z\psi + \partial^2_{xx}\psi + (\tfrac{\omega}{c})^2\varepsilon\big(x, |\psi|^2\big)\psi = 0 \qquad \text{(NLS)}$$

as the governing equation for laser beams propagating in nonlinear inhomogeneous planar waveguides.

Here, $\psi = \psi(x, z) : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ represents the slowly varying envelope of the electric field, z the direction of propagation of the waves, and x the transverse direction.

Nonlinear planar waveguides are key elements in integrated optics — the optical counterpart of integrated electronic circuits.

Self-focusing vs. diffraction

$$i\partial_z\psi + \partial^2_{xx}\psi + (\tfrac{\omega}{c})^2\varepsilon\big(x,|\psi|^2\big)\psi = 0 \qquad \text{(NLS)}$$

The electric permittivity $\varepsilon = \varepsilon(x,|\psi|^2) > 0$ models the (nonlinear) response of the dielectric material to the electromagnetic field carried by the laser beam.

Typically ε is increasing $|\psi|^2 \rightsquigarrow$ self-focusing of the beam.

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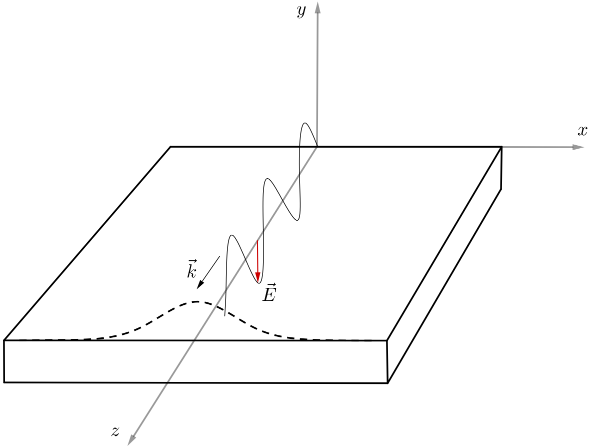
This is due to microscopic changes in the material induced by the electromagnetic field $\rightsquigarrow$ higher (nonlinear) refractive index

$$n(x, |\psi|^2) = \sqrt{\varepsilon(x, |\psi|^2)}.$$

Balance of focusing and diffraction $\rightsquigarrow$ stable localised beams over long distances. We prove rigorously the existence and stability of such coherent nonlinear structures — called solitons.

Planar waveguide



Seek travelling waves $\mathbf{E}(x, z, t) = \mathrm{Re}\left[U(x)e^{i(kz-\omega t)}\right]\mathbf{e}_y$ with $k > 0$ in the optical regime and a localised soliton profile $U > 0$.

Polarised electro-magnetic waves

We seek EM waves corresponding to an electric field of the form

$$\mathbf{E}(x,z,t) = \mathrm{Re}\,[w(x,z)e^{-i\omega t}]\mathbf{e}_y, \qquad\qquad \text{(TE mode)}$$

where $w : \mathbb{R}^2 \to \mathbb{C}$ satisfies the Helmholtz equation

$$\Delta w(x,z) + (\tfrac{\omega}{c})^2 \varepsilon\big(x, |w(x,z)|^2\big) w(x,z) = 0 \quad + \quad \text{G.C.}$$

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The electric permittivity ε can be decomposed as

$$\varepsilon(x,s) = \varepsilon_L(x) + \varepsilon_{NL}(x,s), \quad \text{where we shall suppose}$$

$\varepsilon_L(x) = \varepsilon_0 + \alpha\delta(x)$ for some constants $\varepsilon_0 > 0$, $\alpha \geqslant 0$,
$\varepsilon_{NL}(x,s) > 0$ is increasing in $s \geqslant 0$ and even in $x \in \mathbb{R}$.

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Kerr media have $\varepsilon_{NL}(x, |\psi|^2) = a(x)|\psi|^2 \rightsquigarrow$ cubic nonlinearity.
For saturable media, $\varepsilon_{NL}(x,|\psi|^2) = a(x)\frac{|\psi|^2}{1+|\psi|^2}$ or $2|\psi|^2 - |\psi|^4$.

Travelling waves

$$\Delta w(x,z) + (\tfrac{\omega}{c})^2 \varepsilon\big(x, |w(x,z)|^2\big) w(x,z) = 0$$

We now consider special TE modes in the form of travelling waves

$$w(x,z) = U(x)e^{ikz} \;\leftrightarrow\; \mathbf{E}(x,z,t) = \mathrm{Re}\,[U(x)e^{i(kz-\omega t)}]\mathbf{e}_y.$$

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The soliton profile U must then satisfy the boundary-value problem

$$U''(x) + (\tfrac{\omega}{c})^2 \varepsilon\big(x, U(x)^2\big) U(x) = k^2 U(x), \qquad \text{(TW)}$$
$$U \in H^1(\mathbb{R},\mathbb{R}),\; U(x), U'(x) \to 0 \text{ as } |x| \to \infty.$$

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The power of a beam with profile U_k is given by

$$P = \frac{c^2 k}{2\omega} \int_{-\infty}^{\infty} U_k(x)^2 \,\mathrm{d}x.$$

The paraxial approximation

We are interested in the stability of a travelling wave

$$\mathbf{E}(x, z, t) = \mathrm{Re}\,[U_k(x)e^{i(kz-\omega t)}]\mathbf{e}_y$$

among the set of guided TE modes, with respect to perturbation of the source at $z = 0$,

$$\mathbf{E}(x, 0, t) = U_k(x)\cos(\omega t)\mathbf{e}_y.$$

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We follow a standard approach: consider laser beams collimated along $\mathbf{e}_z$, i.e. with rays almost parallel to $\mathbf{e}_z$, so as to approximate the Helmholtz equation with a nonlinear Schrödinger equation.

To this aim, we shall now consider slow modulations of a plane carrier wave $e^{ik_0 z}$.

Fix some typical wavenumber k_0 of the carrier wave and consider

$$w(x,z) = e^{ik_0 z} W(x,z).$$

Then

$$\Delta w(x,z) = e^{ik_0 z}[\partial_{xx}^2 + \partial_{zz}^2 + 2ik_0\partial_z - k_0^2]W(x,z)$$

or, in Fourier space,

$$[-k_x^2 - (k_z - k_0)^2 - 2k_0(k_z - k_0) - k_0^2]\widehat{W}(k_x, k_z - k_0).$$

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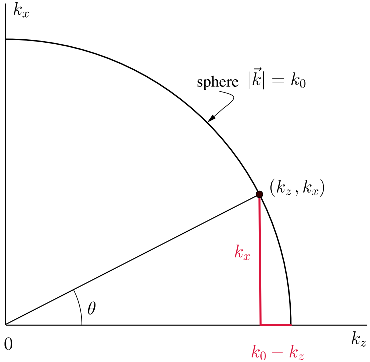
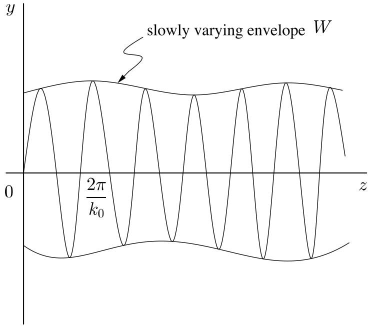
Now, if $\mathbf{k} = (k_x, k_z)$ is almost parallel to $(0, k_0)$, then

$$(k_z - k_0)^2 \ll k_x^2 \quad \text{and} \quad (k_z - k_0)^2 \ll k_0(k_z - k_0)$$

so that

$$\Delta w(x,z) \approx e^{ik_0 z}[\partial^2_{xx} + 2ik_0\partial_z - k_0^2]W(x,z).$$





$$\frac{k_z}{k_0} = \cos\theta = 1 - \frac{\theta^2}{2} + o(\theta^2)$$

$$(k_z - k_0)^2 = -k_0^2 \frac{\theta^4}{4} + o(\theta^4)$$

$$\frac{k_x}{k_0} = \sin\theta = \theta + o(\theta^2)$$

$$k_x^2 \approx k_0^2\theta^2 \gg (k_z - k_0)^2$$

$$|k_z - k_0| \ll k_0 \;\leftrightarrow\; |\partial_z W| \ll k_0 |W|$$

$$|k_z - k_0|^2 \ll k_0^2 \;\leftrightarrow\; |\partial_{zz}^2 W| \ll k_0^2 |W|$$

so W varies little w.r.t. z
over one carrier wavelength $\frac{2\pi}{k_0}$

Hence

$$\begin{aligned} &\partial_{xx}^2 W + \partial_{zz}^2 W + 2ik_0 \partial_z W - k_0^2 W + (\tfrac{\omega}{c})^2 \varepsilon\big(x, |W|^2\big) W = 0 \\ \sim \quad &\partial_{xx}^2 W + 2ik_0 \partial_z W - k_0^2 W + (\tfrac{\omega}{c})^2 \varepsilon\big(x, |W|^2\big) W = 0 \end{aligned}$$

where $$(\tfrac{\omega}{c})^2 \varepsilon(x, s) = (\tfrac{\omega}{c})^2 [\varepsilon_0 + \alpha\delta(x) + \varepsilon_{NL}(x, s)].$$

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where $$(\tfrac{\omega}{c})^2 \varepsilon(x, s) = (\tfrac{\omega}{c})^2 [\varepsilon_0 + \alpha\delta(x) + \varepsilon_{NL}(x, s)].$$

With the change of variables

$$\tau = z/2k_0,\;\; W(x, 2k_0\tau) = e^{i\gamma\tau}\psi(x,\tau),\;\; \gamma = (\tfrac{\omega}{c})^2 \varepsilon_0 - k_0^2$$

we obtain the canonical (dimensionless) form

$$i\partial_\tau \psi + \partial^2_{xx}\psi + (\tfrac{\omega}{c})^2 [\alpha\delta(x) + \varepsilon_{NL}\big(x, |\psi|^2\big)]\psi = 0. \qquad \text{(NLS)}$$

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Now the standing wave $\psi_\lambda(x, \tau) = e^{i\lambda\tau} u_\lambda(x)$ solves (NLS) provided

$$u'' + (\tfrac{\omega}{c})^2 [\alpha\delta(x) + \varepsilon_{NL}\big(x, u^2\big)]u = \lambda u, \qquad \text{(TW)}$$

Furthermore, if $\lambda = k^2 - (\tfrac{\omega}{c})^2 \varepsilon_0$ then $u_\lambda = U_k$.

Part II: Stable solitons via bifurcation

Orbital stability

(NLS) invariant under multiplication by a phase factor $e^{i\theta} \rightsquigarrow$ the appropriate notion of stability is 'orbital stability'.

Definition

We say that the soliton (standing wave) $\psi_\lambda(x,\tau) = e^{i\lambda\tau}u_\lambda(x)$ *is orbitally stable if*

for all $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

for any solution $\varphi(x,\tau)$ *of (NLS) with initial data* $\varphi(\cdot,0) \in H^1(\mathbb{R})$ *there holds*

$$\|\varphi(\cdot,0) - u_\lambda\|_{H^1} \leqslant \delta \implies \inf_{\theta\in\mathbb{R}} \|\varphi(\cdot,z) - e^{i\theta}u_\lambda\|_{H^1} \leqslant \varepsilon \quad \textit{for all } \tau \geqslant 0.$$

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General theory of orbital stability (Grillakis–Shatah–Strauss 1987) $\implies \psi_\lambda$ stable provided:

- spectral conditions on linearised problem
- $\frac{\mathrm{d}}{\mathrm{d}\lambda}\|u_\lambda\|_{L^2}^2 > 0$ (needs $\lambda \mapsto u_\lambda$ to be C^1)

'Cubic' nonlinearity, no linear potential

$\varepsilon(x,s) = \varepsilon_0 + a(x)s, \ a \in C^1(\mathbb{R})$

a even, $a'(x) < 0 \ \forall x > 0, \ a(x) \overset{\infty}{\approx} |x|^{-b}$

$u'' + (\frac{\omega}{c})^2 a(x) u^3 = \lambda u$

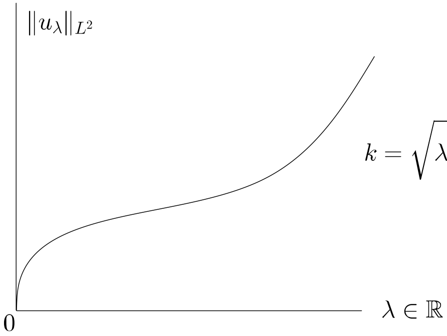
(G.–Stuart 2008, G. 2010)

$\varepsilon(x,s) = \varepsilon_0 + a(x)\dfrac{s}{1+s}$

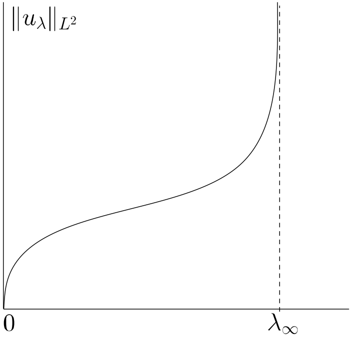
with same $a \in C^1(\mathbb{R})$

$u'' + (\frac{\omega}{c})^2 a(x) \dfrac{u^3}{1+u^2} = \lambda u$

(G. 2013)



$$k = \sqrt{\lambda + \left(\frac{\omega}{c}\right)^2}$$

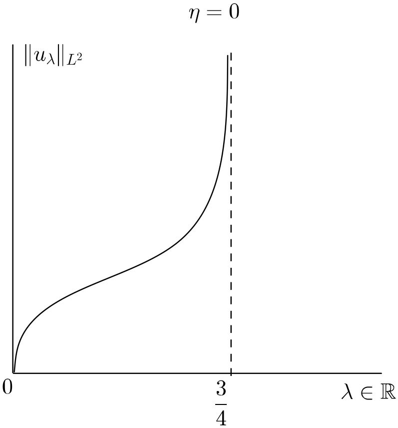


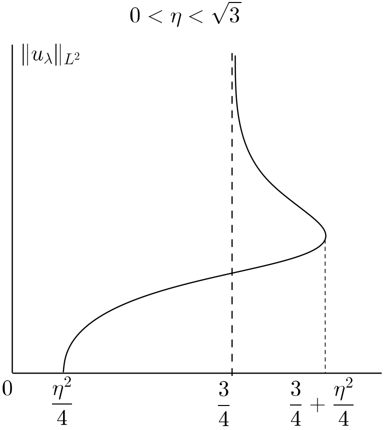
'Cubic–quintic' nonlinearity, delta-function potential

$$\varepsilon(x,s) = \varepsilon_0 + \alpha\delta(x) + \beta s - \gamma s^2, \ \beta,\gamma > 0$$

putting $\eta = (\frac{\omega}{c})^2\alpha, \ 2 = (\frac{\omega}{c})^2\beta, \ 1 = (\frac{\omega}{c})^2\gamma \ \rightsquigarrow$

$u'' + \eta\delta(x) + 2u^3 - u^5 = \lambda u$ (G.–Malomed–Weishäupl 2016)





Explicit solutions of $u'' + \eta\delta(x)u + 2u^3 - u^5 = \lambda u$

For $\eta = 0$ the solutions take the simple form

$$u_\lambda(x) = \sqrt{\frac{2\lambda}{1 + \sqrt{1 - \frac{4\lambda}{3}} \cosh\left(2\sqrt{\lambda}x\right)}}, \quad 0 < \lambda < \tfrac{3}{4}.$$

For $\eta > 0$ the solutions with $\frac{\eta^2}{4} < \lambda < \frac{3}{4}$ are given by

$$u_{\lambda,\eta}(x) =$$
$$\sqrt{\frac{2\lambda}{1 + \frac{\eta + \eta\sqrt{1 + (4\lambda/\eta^2 - 1)(1 - 4\lambda/3)}}{4(\sqrt{\lambda} - \eta/2)} e^{2\sqrt{\lambda}|x|} + \frac{(1 - 4\lambda/3)(\sqrt{\lambda} - \eta/2)}{\eta + \eta\sqrt{1 + (4\lambda/\eta^2 - 1)(1 - 4\lambda/3)}} e^{-2\sqrt{\lambda}|x|}}},$$

which, at $\lambda = \frac{3}{4}$, reduces to

$$u_{\frac{3}{4}}(x) = \sqrt{\frac{3}{2}} \sqrt{\frac{1}{1 + \frac{\eta}{\sqrt{3} - \eta} e^{\sqrt{3}|x|}}}.$$

For $\frac{3}{4} < \lambda < \frac{3}{4} + \frac{\eta^2}{4}$ the expressions are more involved:

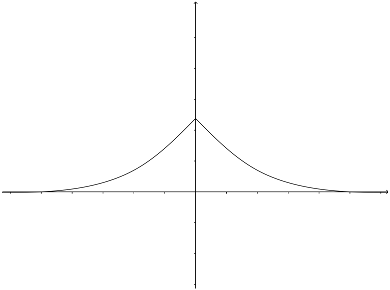
$$u_{\pm,\lambda,\eta}(x) =$$
$$2\sqrt{\frac{\lambda}{\left(e^{\sqrt{\lambda}(|x|-c)} + e^{-\sqrt{\lambda}(|x|-c)}\right)\left(\left(2\sqrt{\frac{\lambda}{3}}+1\right)e^{\sqrt{\lambda}(|x|-c)} - \left(2\sqrt{\frac{\lambda}{3}}-1\right)e^{-\sqrt{\lambda}(|x|-c)}\right)}},$$

where the integration constants $c = c_{\pm,\lambda,\eta} \in \mathbb{R}$ are given by

$$e^{\sqrt{\lambda}\, c_{-,\lambda,\eta}} = \sqrt{\frac{3 - \sqrt{3}\sqrt{3+\eta^2-4\lambda} + 2\eta\sqrt{\lambda} - 4\lambda}{-3 + \sqrt{3}\sqrt{3+\eta^2-4\lambda} + 2\sqrt{3}\sqrt{\lambda} - 2\sqrt{\lambda}\sqrt{3+\eta^2-4\lambda}}}$$

and

$$e^{\sqrt{\lambda}\, c_{+,\lambda,\eta}} = \sqrt{\frac{-3 - \sqrt{3}\sqrt{3+\eta^2-4\lambda} - 2\eta\sqrt{\lambda} + 4\lambda}{3 + \sqrt{3}\sqrt{3+\eta^2-4\lambda} - 2\sqrt{3}\sqrt{\lambda} - 2\sqrt{\lambda}\sqrt{3+\eta^2-4\lambda}}}.$$



At the fold bifurcation point $\lambda = \frac{3}{4} + \frac{\eta^2}{4}$ the solution takes the more tractable form:

$$u_\eta(x) = \sqrt{\frac{3}{2}}\sqrt{\frac{3+\eta^2}{3+\eta^2\cosh(\sqrt{3+\eta^2}|x|)+\eta\sqrt{3+\eta^2}\sinh(\sqrt{3+\eta^2}|x|)}}.$$

THANK YOU!